

Markovian Continuity of the MMSE

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Abstract—Minimum mean square error (MMSE) estimation is widely used in signal processing and related fields. While it is known to be non-continuous with respect to all standard notions of stochastic convergence, it remains robust in practical applications. In this work, we review the known counterexamples to the continuity of the MMSE. We observe that, in these counterexamples, the discontinuity arises from an element in the converging measurement sequence providing more information about the estimand than the limit of the measurement sequence. We argue that this behavior is uncharacteristic of real-world applications and introduce a new stochastic convergence notion, termed Markovian convergence, to address this issue. We prove that the MMSE is, in fact, continuous under this new notion. We supplement this result with semi-continuity and continuity guarantees of the MMSE in other settings and prove the continuity of the MMSE under linear estimation.

Index Terms—Minimum mean square error, estimation error, parameter estimation, inference algorithms, correlation.

I. INTRODUCTION

Minimum mean square error (MMSE) estimation is a cornerstone of statistical signal processing and estimation theory [1]–[3]. Its simple conditional-mean expression, along with its physical interpretation as the minimizer of the mean power of the estimation error, makes it the standard choice for many engineering applications in signal processing, communications, control theory, machine learning, data science, and other domains.

Classic examples of linear MMSE (LMMSE) estimation in signal processing include Wiener and Kalman filters [4]–[7]. These serve as building blocks in control and communications, e.g., in Linear Quadratic Gaussian (LQG) and H^2 control [8], [9], and in feed-forward equalizers (FFE) and decision feedback equalizers (DFE) [10]. The quadratic loss function serves also as a common choice in regression analysis [11], [12] and machine learning (ML), and is a common choice in reinforcement [13] and online learning [14].

While many of the above solutions are linear, non-linear variants thereof exist relying both on classical [15]–[17] and modern ML-based techniques [18]–[21].

These techniques rely primarily on models that rely on statistical knowledge. As the required statistics is acquired from finite samples and finite-precision/noisy measurements, continuity of the MMSE and the corresponding estimators is implicitly assumed. Similar implicit assumption are characteristic also of model-free reinforcement learning [13].

However, despite being considered robust in practice, the MMSE is known to be non-continuous in general.

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Namely, consider a sequence of pairs of random variables $\{(X_n, Y_n)\}_{n=1}^{\infty}$ converging in some standard stochastic sense (in distribution, in probability, in mean square, almost surely) to a pair of random variables (X, Y) . Here, (X_n, Y_n) can represent, e.g., an empirical distribution resulting from a finite sample of length n drawn from the distribution of (X, Y) , or a finite-precision variant of (X, Y) with the machine precision increasing with n . Then,

$$\text{MMSE}(X_n|Y_n) \not\rightarrow \text{MMSE}(X|Y)$$

in general, where $\text{MMSE}(X|Y)$ denotes the MMSE in estimating the random parameter X from the measurement Y . As is common, we will say that the MMSE is continuous/discontinuous in a particular stochastic sense depending on the stochastic convergence sense of the sequence $\{(X_n, Y_n)\}_{n=1}^{\infty}$ to (X, Y) .

Wu and Verdú [22], and Yüksel and Linder [23] (see also [24], [25, Chapter 8.3]) provided concrete counterexamples to the continuity of the MMSE, demonstrating that the discontinuity may even be unbounded.

Wu and Verdú [22] proved that the MMSE is upper semi-continuous (u.s.c.) in distribution, as long as $\{X_n\}_{n=1}^{\infty}$ and X are uniformly bounded. For additive channels with independent noise N in X and $\{X_n\}_{n=1}^{\infty}$ where X and N have finite second moments (finite power), they proved that the MMSE is also u.s.c. in distribution. Further, if N has a bounded continuous probability density function (PDF), then the MMSE is also continuous in distribution, to wit

$$\lim_{n \rightarrow \infty} \text{MMSE}(X_n|X_n + N) = \text{MMSE}(X|X + N).$$

Yüksel and Linder [23] (see also [25, Chapter 8.3]) considered the case of a common parameter $X = X_n$ for all n , and a sequence in n of channels $\{P_{Y_n|X}\}_{n=1}^{\infty}$ from X to Y_n , that converges to a channel $P_{Y|X}$ from X to Y . They proved that, for bounded and continuous distortion measures between X and its estimate, the MMSE is u.s.c. in distribution. Hogeboom-Burr and Yüksel [26], [27] (see also [25, Chapter 8.3]) strengthened the u.s.c. in distribution guarantee to a continuity one by restricting the sequence of channels to be *stochastically degraded/garbled* [25, Chapter 7.3], depicted in Figure 1, satisfying:

- a) $P_{Y_n|X}$ is stochastically degraded with respect to $P_{Y|X}$.
- b) $P_{Y_n|X}$ is stochastically degraded with respect to $P_{Y_{n+1}|X}$.

Unfortunately, these continuity results are limited to specific settings and do not fully explain the robustness of the MMSE which is observed in practical scenarios.

In this work, we establish new continuity results for the MMSE, which subsume the aforementioned results. To that

end, we first review existing counterexamples to and guarantees of the continuity of the MMSE in Section II. We identify common traits in these counterexamples: For a sequence of pairs of random variables $\{(X_n, Y_n)\}$ converges to (X, Y) in probability, $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{p} (X, Y)$, either

- the second moment (mean power) of X_n does not converge to that of X , viz. $E[X_n^2] \not\rightarrow E[X^2]$,

or

- the MMSE in estimating X from Y_n is strictly better than the MMSE of estimating X from Y .

We argue that such behavior is uncharacteristic of real-world applications and suggest adding two additional requirements:

- 1) convergence of the second moment:

$$\lim_{n \rightarrow \infty} E[X_n^2] = E[X^2];$$

- 2) a Markovian restriction

$$X \circlearrowleft Y \circlearrowleft Y_n \quad (1)$$

for all n , depicted in Figure 2. This restriction amounts to assuming that, given X , Y_n is degraded with respect to Y .

We prove in Section III that, under these two restrictions, the MMSE is *Markov continuous* in probability. Byproducts of this result include continuity guarantees of finite-power additive noises with diminishing power and rounding errors with increasing machine precision.

In Section IV, we prove that the MMSE is u.s.c. in distribution as long as the second moment of X_n converges to that of X (requirement 1 above). This requirement is weaker than the uniform boundness requirement of [22]. For the case of a common parameter $X = X_n$ with a finite second moment, and a sequence $\{P_{Y_n|X}\}_{n=1}^{\infty}$ of channels converging to a channel $P_{Y|X}$, we further show that the MMSE is continuous in distribution as long as $\{P_{Y_n|X}\}_{n=1}^{\infty}$ satisfies the degradedness requirement a above (see Figure 2). In particular, this means that requirement b above is superfluous.

In Section V, we supplement the above results by proving that the MMSE under linear estimation (LMMSE) is continuous in distribution as long as the second moments of $\{X_n\}_{n=1}^{\infty}$ and of $\{Y_n\}_{n=1}^{\infty}$ converge to those of X and Y , respectively.

We establish all the results in this work in the more general framework of random vector (RV) parameters and measurements.

We conclude the paper with Section VI by a summary and discussion of possible future directions.

Next, we introduce the notation used in this paper; necessary background about stochastic convergence and stochastic degradedness is provided in Appendix A.

A. Notation

$\mathbb{R}, \mathbb{N}, \mathbb{Z}$	The sets of real, natural (positive integer), and integer numbers, respectively.
$x[i]$	The i^{th} entry of vector $x \in \mathbb{R}^k$ for $i \in \{1, 2, \dots, k\}$.

x^T	The transpose of a vector x .
x^2	$(x^2[1], x^2[2], \dots, x^2[k])^T$ for $x \in \mathbb{R}^k$ and $k \in \mathbb{N}$ (entrywise squaring).
$\langle x, y \rangle$	$x^T y$ —the standard Euclidean inner product between vectors $x, y \in \mathbb{R}^k$ for $k \in \mathbb{N}$.
$ x $	The absolute value of $x \in \mathbb{R}$.
$\ x\ $	$\sqrt{\langle x, x \rangle}$ —the standard Euclidean norm of a vector x .
$x \leq y$	$x[i] \leq y[i]$ for all $i \in \{1, 2, \dots, k\}$, where $x, y \in \mathbb{R}^k$ and $k \in \mathbb{N}$.
$\lfloor x \rfloor$	The floor operation applied entrywise to the entries of $x \in \mathbb{R}^k$ for $k \in \mathbb{N}$.
$\text{sign}\{x\}$	The sign of $x \in \mathbb{R}$.
$\text{trace}\{A\}$	The trace of $A^{k \times k} \in \mathbb{R}$ for $k \in \mathbb{N}$.
$\lim, \overline{\lim}, \underline{\lim}$	Limit, limit superior, and limit inferior, respectively.
E, Var	Expectation and variance operators, respectively.
$\langle X, Y \rangle_{\text{RV}}$	$E[\langle X, Y \rangle] = E[X^T Y]$ for random vectors (RVs) $X, Y \in \mathbb{R}^k$ where $k \in \mathbb{N}$.
$\ X\ _{\text{RV}}$	$\sqrt{\langle X, X \rangle_{\text{RV}}} = \sqrt{E[X^T X]}$ for an RV $X \in \mathbb{R}^k$ where $k \in \mathbb{N}$.
$X \stackrel{d}{=} Y$	X and Y have the same distribution.
$X \perp\!\!\!\perp Y$	Independence between RVs X and Y .
$X \circlearrowleft Y \circlearrowleft Z$	Markov triplet: X and Z are independent given Y .
$X \stackrel{d}{\circlearrowleft} Y \stackrel{d}{\circlearrowleft} Z$	Garbled triplet: the conditional distribution of Z given X is stochastically degraded/garbled with respect to the conditional distribution of Z given X , with respect to X (see also Definition A.5).
$X_n \xrightarrow[n \rightarrow \infty]{d} X$	Convergence in distribution of $\{X_n\}_{n=1}^{\infty}$ to X .
$X_n \xrightarrow[n \rightarrow \infty]{p} X$	Convergence in probability of $\{X_n\}_{n=1}^{\infty}$ to X .
$X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$	Almost-sure convergence of $\{X_n\}_{n=1}^{\infty}$ to X .
$X_n \xrightarrow[n \rightarrow \infty]{m.s.} X$	Mean-square convergence of $\{X_n\}_{n=1}^{\infty}$ to X .
$\text{MMSE}(X Y)$	The MMSE in estimating X given Y .
$\text{LMMSE}(X Y)$	The LMMSE in (linearly) estimating X given Y .

II. DISCUSSION OF EXISTING RESULTS

We first present the definition of the MMSE [28, Chapter 8], [3, Chapter 4], [29, Chapter 7].

Definition II.1. The MMSE in estimating an RV X with a finite second moment, $\|X\|_{\text{RV}} < \infty$, from an RV Y is defined as

$$\text{MMSE}(X|Y) \triangleq \inf \|X - \hat{X}\|_{\text{RV}}^2,$$

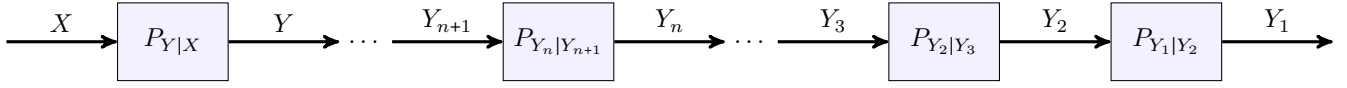


Fig. 1. Illustration of a nested sequence of (physically) degraded channels $X \rightarrow Y \rightarrow \dots \rightarrow Y_{n+1} \rightarrow Y_n \rightarrow \dots \rightarrow Y_3 \rightarrow Y_2 \rightarrow Y_1$.

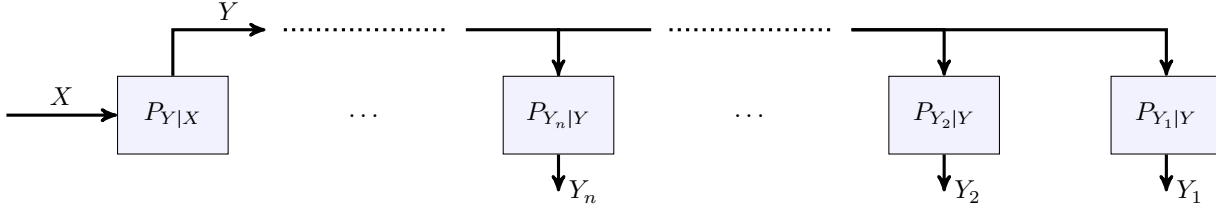


Fig. 2. Illustration of a sequence of individually (physically) degraded channels: $X \rightarrow Y \rightarrow Y_i$ for all $i \in \{1, 2, \dots, n\}$. This is a less stringent requirement than the one depicted in Figure 1 as it does not assume degradedness between X_i and X_j for $i \neq j$.

where the infimum is over all RVs \hat{X} with finite second moment that satisfy $X \rightarrow Y \rightarrow \hat{X}$.

The following is a known characterization of the MMSE [3, Chapter 4], [30, Chapter 9.1.5], [7, Appendix for Chapter 3] which is often used as its definition.

Theorem II.1. *The MMSE estimate of an RV X with a finite second moment, $\|X\|_{\text{RV}} < \infty$, from an RV Y is given by $E[X|Y]$, and the corresponding MMSE is given as*

$$\text{MMSE}(X|Y) = \|X - E[X|Y]\|_{\text{RV}}^2 \quad (2a)$$

$$= \|X\|_{\text{RV}}^2 - \|E[X|Y]\|_{\text{RV}}^2. \quad (2b)$$

It is well known that the MMSE is not continuous in general [22]–[24], [25, Chapter 8.3]. We start by recalling known counterexamples that demonstrate it.

We first demonstrate that even in the absence of measurements, the MMSE which reduces to the variance, might not be continuous.

Example II.1. Let $Y = Y_n = 0$ for all $n \in \mathbb{N}$. Set $X = 0$ and

$$X_n = \begin{cases} \sqrt{n}, & \text{w.p. } \frac{1}{2n} \\ -\sqrt{n}, & \text{w.p. } \frac{1}{2n} \\ 0, & \text{w.p. } 1 - \frac{1}{n}. \end{cases}$$

Clearly, $X_n \xrightarrow{d} 0 = X$, but

$$\lim_{n \rightarrow \infty} \|X_n - X\| = \lim_{n \rightarrow \infty} \|X_n\| = \lim_{n \rightarrow \infty} 1 = 1,$$

meaning that $X_n \xrightarrow{m.s.} X$ and

$$\lim_{n \rightarrow \infty} E[X_n^2] = 1 > 0 = E[X^2].$$

Consequently, for all $n \in \mathbb{N}$,

$$\text{MMSE}(X_n|Y_n) = \|X_n\|_{\text{RV}}^2 = 1, > 0 = \text{MMSE}(X|Y)$$

meaning that the MMSE is not continuous in this case:

$$\lim_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) = 1 > 0 = \text{MMSE}(X|Y).$$

The latter further suggests that, in this example, the MMSE is lower semi-continuous (l.s.c.) but not u.s.c.

Even when $X_n \xrightarrow{m.s.} X$, the MMSE might not be continuous when a Markovian restriction of the form (1) does not hold. This is demonstrated in the following two examples.

Example II.2. Let X and Y be independent random variables such that X has bounded support and $Y \in \mathbb{Z}$ with $\|Y\|_{\text{RV}} < \infty$. For concreteness, let Y be equiprobable Bernoulli distributed, and let X be uniformly distributed over the unit interval. Define $X_n = X$ and

$$Y_n = Y + \frac{X}{n}$$

for all $n \in \mathbb{N}$.

Since $X_n = X$ for all $n \in \mathbb{N}$ and $\|X\|_{\text{RV}} < \infty$, $X_n \xrightarrow{m.s.} X$ and $X_n \xrightarrow{a.s.} X$ trivially hold.

Since X is bounded, $\|X\|_{\text{RV}} < \infty$. Consequently,

$$\|Y_n\|_{\text{RV}} = \left\| Y + \frac{X}{n} \right\|_{\text{RV}} \leq \|Y\|_{\text{RV}} + \frac{1}{n} \|X\|_{\text{RV}} < \infty.$$

Furthermore,

$$\lim_{n \rightarrow \infty} \|Y_n - Y\|_{\text{RV}} = \lim_{n \rightarrow \infty} \left\| \frac{X}{n} \right\|_{\text{RV}} = 0,$$

Hence, $Y_n \xrightarrow{m.s.} Y$. Furthermore, $\{Y_n\}_{n=1}^{\infty} \xrightarrow{a.s.} Y$.

Since $X \perp\!\!\!\perp Y$, $\text{MMSE}(X|Y) = \text{Var}(X) = 1/12$.

However, since X can be perfectly estimated from Y_n from its fractional part, viz. $X = n(Y_n - \lfloor Y_n \rfloor)$ a.s.,

$$\text{MMSE}(X_n|Y_n) = \text{MMSE}(X|Y_n) = 0 \quad \forall n \in \mathbb{N}.$$

Hence, the MMSE is not continuous in this example:

$$\lim_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) = 0 < \frac{1}{12} = \text{MMSE}(X|Y).$$

In particular, the MMSE is u.s.c. but not l.s.c. in this example. Note further that the Markovian relation (1) does not hold in this example.

Example II.3. Let X and N be independent Rademacher RVs, and $Y = X + N$. In particular, $\text{Var}(X) = 1 < \infty$. Let $X_n = \frac{n}{n+1}X$ and $Y_n = X_n + N$ for all $n \in \mathbb{N}$.

Clearly, $(X_n, Y_n) \xrightarrow{a.s.} (X, Y)$ and

$$\lim_{n \rightarrow \infty} E[X_n^2] = \lim_{n \rightarrow \infty} \frac{n}{n+1} E[X^2] = E[X^2],$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \mathbb{E}[Y_n^2] &= \lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] + \mathbb{E}[N^2] \\ &= \mathbb{E}[X^2] + \mathbb{E}[N^2] = \mathbb{E}[Y^2].\end{aligned}$$

Hence $(X_n, Y_n) \xrightarrow[m.s.]{n \rightarrow \infty} (X, Y)$ (see Theorems A.1 and A.2).

Since X_n can be perfectly estimated from Y_n from its fractional part, viz.. $X_n = Y_n - \text{sign}\{Y_n\}$ a.s., $\text{MMSE}(X_n|Y_n) = 0$ for all $n \in \mathbb{N}$. However, X cannot be perfectly estimated from $Y = X + N$. In fact, $\mathbb{E}[X|Y] = Y/2$ and

$$\text{MMSE}(X|Y) = \mathbb{E}\left[\left(\frac{X - N}{2}\right)^2\right] = \frac{1}{2}.$$

Hence, the MMSE is not continuous:

$$\lim_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) = 0 < 1/2 = \text{MMSE}(X|Y).$$

Again, the latter suggests that the MMSE is u.s.c. but not l.s.c. in this example. And again, we note that the Markovian relation (1) does not hold in this example.

While MMSE is not generally continuous or even semi-continuous in general, it was proved by Wu and Verdú [22, Theorem 3] to be u.s.c. if the supports of the RVs X and $\{X_n\}_{n=1}^{\infty}$ are uniformly bounded.

Theorem II.2 ([22, Theorem 3]). *Let (X, Y) be a pair of RVs and let $\{(X_n, Y_n)\}_{n=1}^{\infty}$ be a sequence of pairs of RVs such that*

- $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y)$;
- $\mathbb{P}(\|X\| \leq m) = 1$ and $\mathbb{P}(\|X_n\| \leq m) = 1$ for all $n \in \mathbb{N}$ for some $m \in \mathbb{R}$.

Then, the MMSE is u.s.c. in distribution:

$$\overline{\lim}_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) \leq \text{MMSE}(X|Y). \quad (3)$$

When restricting the possible statistical relations, the following continuity results have been proved.

Theorem II.3 ([22, Theorem 4]). *Let X and N be a pair of RVs of the same length $\{(X_n, Y_n)\}_{n=1}^{\infty}$ be a sequence of pairs of RVs, such that*

- $Y = X + N$, and $Y_n = X_n + N$ for all $n \in \mathbb{N}$;
- $N \perp\!\!\!\perp X, \{X_n\}_{n=1}^{\infty}$;
- $\|X\|_{\text{RV}}, \|N\|_{\text{RV}} < \infty$;
- $X_n \xrightarrow[n \rightarrow \infty]{d} X$.

Then,

- *the MMSE is u.s.c. (3) in distribution.*
- *In addition, if N has a probability density function (PDF) that is bounded and continuous, then the MMSE is continuous in distribution:*

$$\lim_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) = \text{MMSE}(X|Y).$$

Unfortunately, the result above is limited to additive noise channels where the noise N has bounded and continuous PDF. In particular, it does not guarantee MMSE continuity, e.g., for noises with continuous uniform or arcsine distributions, or noises whose distribution contains discrete or singular behavior (recall Lebesgue's decomposition theorem [31, Chapter 2, Section 2.3]).

Remark II.1. Furthermore, as indicated in [22], since

$$\begin{aligned}\text{MMSE}(X_n|Y_n) &= \inf_{g_1} \|X_n - g_1(Y_n)\|_{\text{RV}}^2 \\ &= \inf_{g_1} \|X_n - Y_n + Y_n - g_1(Y_n)\|_{\text{RV}}^2 \\ &= \inf_{g_2} \|N - g_2(Y_n)\|_{\text{RV}}^2 \\ &= \text{MMSE}(N|Y_n),\end{aligned}$$

the setting of Theorem II.3 can be viewed as an estimation problem of a fixed parameter N from Y_n , where Y_n is the output of an additive noise channel with noise X_n , and where $Y_n \xrightarrow[n \rightarrow \infty]{d} Y$. Since $N \perp\!\!\!\perp X, \{X_n\}_{n=1}^{\infty}$, this means further that

$$(N, X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (N, X, Y).$$

The framework where a common parameter X passes through a sequence of channels resulting in a sequence $\{X_n\}_{n=1}^{\infty}$ was also studied by Yüksel and Linder [23, Theorem 3.2] and Hogeboom-Burr and Yüksel [26], [27] (see also [25, Chapter 8.3]). The following theorem and remarks summarize relevant results about continuity in distribution.

Theorem II.4 ([27], [25, Theorem 8.3.4]). *Let $c : \mathcal{X} \times \hat{\mathcal{X}} \rightarrow \mathbb{R}$, $X \in \mathcal{X}$ be an RV, and $\{Y_n \in \mathcal{Y}\}_{n=1}^{\infty}$ be a sequence of RVs such that*

- 1) *c is continuous and bounded;*
- 2) *\mathcal{X} is a convex set;*
- 3) $(X, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y)$;
- 4) $X \xrightarrow{d} Y \xrightarrow{d} Y_n$ for all $n \in \mathbb{N}$;
- 5) $X \xrightarrow{d} Y_{n+1} \xrightarrow{d} Y_n$ for all $n \in \mathbb{N}$.

Then,

$$\lim_{n \rightarrow \infty} \inf_{g: \mathcal{Y} \rightarrow \hat{\mathcal{X}}} \mathbb{E}[c(X, g(Y_n))] = \inf_{g: \mathcal{Y} \rightarrow \hat{\mathcal{X}}} \mathbb{E}[c(X, g(Y))].$$

Requirement (4), $X \xrightarrow{d} Y \xrightarrow{d} Y_n$, means that the channel from X to Y_n is stochastically degraded with respect to the channel from X to Y . Equivalently, this requirement means that there exist probabilistically identical channels to these two channels such that, for the same input X , their outputs satisfy (1). See Appendix A for further details about stochastic degradedness.

Remark II.2. For uniformly bounded RVs X, Y , and $\{Y_n\}_{n=1}^{\infty}$, Theorem II.4 can be readily applied to the MMSE by selecting a quadratic cost function.

Remark II.3. When only requirements 1–3 hold, Yüksel and Linder [23, Theorem 3.2] (see also [25, Theorem 8.3.3]) proved that u.s.c. in distribution holds:

$$\overline{\lim}_{n \rightarrow \infty} \inf_{g: \mathcal{Y} \rightarrow \hat{\mathcal{X}}} \mathbb{E}[c(X, g(Y_n))] = \inf_{g: \mathcal{Y} \rightarrow \hat{\mathcal{X}}} \mathbb{E}[c(X, g(Y))].$$

This is subsumed by the result of Theorem II.2 for a quadratic cost function c and bounded RVs.

While Theorem II.4 extends the continuity guarantees beyond the scope of additive noise channels of Theorem II.3, it

is limited to bounded RVs and *nested* garbling of $\{Y_n\}_{n=1}^\infty$ and Y (see also Figure 1):

$$X \overset{d}{\dashrightarrow} Y \overset{d}{\dashrightarrow} \dots \overset{d}{\dashrightarrow} Y_{n+1} \overset{d}{\dashrightarrow} \dots \overset{d}{\dashrightarrow} Y_2 \overset{d}{\dashrightarrow} Y_1. \quad (4)$$

This is demonstrated by the following additive-noise example.

Example II.4. Let X and N be independent continuous random variables uniformly distributed over the interval $[-\sqrt{3}, \sqrt{3}]$. Let $Y_n = X + N/n$ and $Y = X$. Note that the second moments are all finite: $\|X\|_{\text{RV}} = \|Y\|_{\text{RV}} = 1$ and $\|Y_n\| = 1 + n^{-2} \leq 2$ for all $n \in \mathbb{N}$. Furthermore,

$$\lim_{n \rightarrow \infty} \|Y_n - Y\|_{\text{RV}} = \lim_{n \rightarrow \infty} \frac{\|N\|_{\text{RV}}}{n} = 0.$$

Therefore, $Y_n \xrightarrow{m.s.} Y = X$, meaning that

$$\lim_{n \rightarrow \infty} \text{MMSE}(X|Y_n) = 0 = \text{MMSE}(X|Y)$$

by the squeeze theorem:

$$0 \leq \lim_{n \rightarrow \infty} \text{MMSE}(X|Y_n) \leq \lim_{n \rightarrow \infty} \|X - Y_n\|_{\text{RV}} = 0.$$

However, since requirement 5 in Theorem II.4 does not hold for any $n \in \mathbb{N}$, this theorem cannot be applied for this case. The conditions of Theorem II.3 (recall Remark II.1) do not hold either since the uniform distribution is not continuous at its support boundaries.

Remark II.4. Replacing the converging noise sequence $\{N/n\}_{n=1}^\infty$ with certain other converging uniform noise sequences, e.g., $\{N/2^n\}_{n=1}^\infty$, may satisfy (4). However, taking the distribution of N to be triangular or arcsine would violate (4). The latter choice also violates the boundedness condition of Theorem II.3.

While these results provide guarantees for the continuity of the MMSE for certain cases, their scope remains limited. In Sections III and IV, we provide guarantees for the continuity of the MMSE under a larger framework. We further supplement these results by establishing continuity in distribution of the MMSE under linear estimation in Section V.

III. MMSE MARKOVIAN CONTINUITY IN PROBABILITY

In practical scenarios, the deviation of Y_n from Y does not carry extra information about the nominal parameter X beyond the information provided by the measurement Y . More precisely, a Markov relation $X \dashrightarrow Y \dashrightarrow Y_n$, as in (1), holds for all $n \in \mathbb{N}$. This Markovian restriction, which is depicted in Figure 2, excludes Examples II.2 and II.3 but holds for Example II.4. We therefore propose the following new sense of stochastic convergence.

Definition III.1. A sequence of pairs of RVs $\{(X_n, Y_n)\}_{n=1}^\infty$ *Markov converges* in probability to a pair of RVs (X, Y) if

$$\begin{aligned} X_n &\xrightarrow[n \rightarrow \infty]{p} X, \\ Y_n &\xrightarrow[n \rightarrow \infty]{p} Y, \end{aligned} \quad (5)$$

and the Markov relation (1) holds for all $n \in \mathbb{N}$. We denote this convergence by $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{M.p.} (X, Y)$.

Remark III.1. The separate convergences in (5) are equivalent to (see Lemma A.1)

$$(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{p} (X, Y)$$

as long as X_n and X (and hence also Y_n and Y) are of the same size for all $n \in \mathbb{N}$. However, while $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{p} (X, Y)$ is equivalent to $(Y_n, X_n) \xrightarrow[n \rightarrow \infty]{p} (Y, X)$, $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{M.p.} (X, Y)$ and $(Y_n, X_n) \xrightarrow[n \rightarrow \infty]{M.p.} (Y, X)$ are not equivalent.

Remark III.2. Markovian variants of a.s. and m.s. convergences can be similarly defined. Viewing this problem in communication-channel terms, to define proper Markovian ordering, a common input X needs to be assumed to result in channel outputs Y and $\{Y_n\}_{n=1}^\infty$ that satisfy the Markovian relation (1). Hence, convergence in probability is assumed in Definition III.1 in order to relate the channel from X to Y to that from X_n to Y_n . In case of a common parameter,

$$X \stackrel{d}{=} X_1 \stackrel{d}{=} X_2 \stackrel{d}{=} \dots \stackrel{d}{=} X_n,$$

the convergence in probability can be replaced with a convergence in distribution with the Markovity property replaced by stochastic degradedness; see Section IV.

Remark III.3. Under the Markovian restriction (1), Y_n may still carry extra information about X_n (but not on X) beyond that of Y . For example, the case of the same diminishing noise which is added to both Y and X satisfies (1):

$$\begin{aligned} X_n &= X + \frac{Z}{n}, \\ Y_n &= Y + \frac{Z}{n}, \end{aligned}$$

for all $n \in \mathbb{N}$ with $Z \perp\!\!\!\perp (X, Y)$; see Corollary III.1 in the sequel.

The following theorem, proved in the sequel, states that, under the Markovian restriction (1) and assuming a converging second moment of the parameter, the MMSE is continuous. We term such continuity *Markovian continuity*.

Theorem III.1. *Let (X, Y) be a pair of RVs and let $\{(X_n, Y_n)\}_{n=1}^\infty$ be a sequence of pairs of RVs such that*

$$(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{M.p.} (X, Y),$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2]. \quad (6)$$

Then, $\mathbb{E}[X_n|Y_n] \xrightarrow[n \rightarrow \infty]{m.s.} \mathbb{E}[X|Y]$ and the MMSE is Markov continuous in probability:

$$\lim_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) = \text{MMSE}(X|Y).$$

Remark III.4. Condition (6) can be replaced by $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X$ or by $\{X_n^2\}_{n=1}^\infty$ being uniformly integrable (u.i.); see Theorem A.2 for details.

Remark III.5. When Y is deterministic, the Markov condition (1) reduces to $Y_n \perp\!\!\!\perp X$ for all $n \in \mathbb{N}$. This is the case in Example II.2 with $Y = 0$.

Sequences that comply with the Markov restriction (1) and the converging second moment restriction of Theorem III.1 include corruption by independent additive noises of decreasing strength and floating point representations with increasing machine precision. This is summarized in the following two corollaries, which are proved in Appendix E.

Corollary III.1 (Additive noise effect). *Let X, M, N be RVs such that $\|X\|_{\text{RV}}, \|N\|_{\text{RV}} < \infty$; $M \perp (X, Y)$; $X, N \in \mathbb{R}^k$ for $k \in \mathbb{N}$; and $Y, M \in \mathbb{R}^m$ for $m \in \mathbb{N}$. Then,*

$$\lim_{(\lambda, \gamma) \rightarrow (0, 0)} \text{MMSE}(X + \gamma N | Y + \lambda M) = \text{MMSE}(X | Y). \quad (7)$$

Corollary III.2 (Machine percision effect). *Let X and Y be two RVs such that $\|X\|_{\text{RV}} < \infty$. Define $[x]_a \triangleq [x/a] \cdot a$. Then,*

$$\lim_{(\lambda, \gamma) \rightarrow (0, 0)} \text{MMSE}([X]_\gamma | [Y]_\lambda) = \text{MMSE}(X | Y).$$

To prove Theorem III.1, we will first prove two special cases, which are of interest on their own right.

Lemma III.1. *Let $\{(X_n, Y_n)\}_{n=1}^\infty$ be a sequence of pairs of RVs and let X_∞ be an RV such that $\|X_\infty\|_{\text{RV}} < \infty$ and $X_n \xrightarrow{m.s.} X_\infty$. Then,*

$$\mathbb{E}[X_n | Y_n] \xrightarrow{m.s.} \mathbb{E}[X_\infty | Y_n].$$

Proof: Denote $\hat{X}_{\infty|n} \triangleq \mathbb{E}[X_\infty | Y_n]$, $\hat{X}_{n|n} \triangleq \mathbb{E}[X_n | Y_n]$, $\check{X}_{\infty|n} \triangleq X_\infty - \hat{X}_{\infty|n}$, $\check{X}_{n|n} \triangleq X_n - \hat{X}_{n|n}$. Then,

$$\begin{aligned} \|X_\infty - X_n\|_{\text{RV}}^2 &= \|\hat{X}_{\infty|n} - \hat{X}_{n|n}\|_{\text{RV}}^2 + \|\check{X}_{\infty|n} - \check{X}_{n|n}\|_{\text{RV}}^2 \\ &\quad + 2 \langle \hat{X}_{\infty|n} - \hat{X}_{n|n}, \check{X}_{\infty|n} - \check{X}_{n|n} \rangle_{\text{RV}} \\ &= \|\hat{X}_{\infty|n} - \hat{X}_{n|n}\|_{\text{RV}}^2 + \|\check{X}_{\infty|n} - \check{X}_{n|n}\|_{\text{RV}}^2, \end{aligned}$$

where the second step follows from the orthogonality principle of MMSE estimation [30, Chapter 9.1.5], [28, Chapter 8.6].

Since $X_n \xrightarrow{m.s.} X_\infty$, both $\lim_{n \rightarrow \infty} \|\hat{X}_{n|n} - \hat{X}_{\infty|n}\|_{\text{RV}} = 0$ and $\lim_{n \rightarrow \infty} \|\check{X}_{n|n} - \check{X}_{\infty|n}\|_{\text{RV}} = 0$. ■

The proof of the following lemma is available in Appendix C.

Lemma III.2. *Let X and Y be two RVs such that $\|X\|_{\text{RV}} < \infty$. Let $\{Y_n\}_{n=1}^\infty$ be a sequence of RVs such that $(X, Y_n) \xrightarrow{M.p.} (X, Y)$. Then,*

$$\mathbb{E}[X | Y_n] \xrightarrow{m.s.} \mathbb{E}[X | Y].$$

We are now ready to prove Theorem III.1.

Proof of Theorem III.1: Let $\epsilon > 0$, however small. By (6) and since $X_n \xrightarrow{p} X$, we have $X_n \xrightarrow{m.s.} X$ (see Theorem A.2). Then, by lemmata III.1 and III.2, there exists $n_0 \in \mathbb{N}$, such that, for all $n > n_0$,

$$\begin{aligned} \|\mathbb{E}[X | Y_n] - \mathbb{E}[X_n | Y_n]\|_{\text{RV}} &< \epsilon, \\ \|\mathbb{E}[X | Y] - \mathbb{E}[X | Y_n]\|_{\text{RV}} &< \epsilon. \end{aligned}$$

Hence, by the triangle (Minkowski) inequality,

$$\begin{aligned} \|\mathbb{E}[X | Y] - \mathbb{E}[X_n | Y_n]\|_{\text{RV}} &\leq \|\mathbb{E}[X | Y_n] - \mathbb{E}[X_n | Y_n]\|_{\text{RV}} \\ &\quad + \|\mathbb{E}[X | Y] - \mathbb{E}[X | Y_n]\|_{\text{RV}} \end{aligned}$$

$< 2\epsilon$.

Since $\epsilon > 0$ is arbitrary, $\mathbb{E}[X_n | Y_n] \xrightarrow{m.s.} \mathbb{E}[X | Y]$. Since m.s. convergence guarantees convergence of the second moment (see by Theorem A.2), we further have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\mathbb{E}[X_n | Y_n]\|_{\text{RV}} &= \|\mathbb{E}[X | Y]\|_{\text{RV}}, \\ \lim_{n \rightarrow \infty} \|X_n\|_{\text{RV}} &= \|X\|_{\text{RV}}. \end{aligned}$$

Then, using the standard formula of the MMSE (see Theorem II.1), we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{MMSE}(X_n | Y_n) &= \lim_{n \rightarrow \infty} (\|X_n\|_{\text{RV}}^2 - \|\mathbb{E}[X_n | Y_n]\|_{\text{RV}}^2) \\ &= \|X\|_{\text{RV}}^2 - \|\mathbb{E}[X | Y]\|_{\text{RV}}^2 \\ &= \text{MMSE}(X | Y), \end{aligned}$$

which concludes the proof. ■

IV. MMSE CONTINUITY IN DISTRIBUTION

In this section, we present results regarding continuity properties in distribution of the MMSE.

We first present a result about the semi-continuity in distribution of the MMSE, which replaces the bounded-support requirement of Theorem II.2 by a relaxed requirement of convergence of the second moment.

Theorem IV.1. *Let (X, Y) be a pair of RVs and let $\{(X_n, Y_n)\}_{n=1}^\infty$ be a sequence of pairs of RVs such that*

- $(X_n, Y_n) \xrightarrow{d} (X, Y)$;
- $\lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2]$.

Then, the MMSE is u.s.c. in distribution:

$$\overline{\lim}_{n \rightarrow \infty} \text{MMSE}(X_n | Y_n) \leq \text{MMSE}(X | Y).$$

The proof of Theorem IV.1 is available in Appendix D.

Recalling Remark III.2, for the case of a common parameter distribution, the *Markovity* requirement and the *convergence in probability* requirement of Theorem III.1 can be replaced with a *stochastic degradedness* requirement and a *convergence in distribution* requirement. This is stated in the following theorem, which implies that requirement 5 of Theorem II.4, $X \xrightarrow{d} Y_{n+1} \xrightarrow{d} Y_n$, is not necessary for the contiuity of the MMSE as long as requirement 5, $X \xrightarrow{d} Y \xrightarrow{d} Y_n$, continues to hold. Namely, the nested garbling requirement of Figure 1 may be replaced by the individual garbling requirement depicted in Figure 2. Since we focus on the MMSE, the cost function is taken to be quadratic: $c(x, \hat{x}) = (x, \hat{x})^2$. We further note that the requirement of bounded RVs of Remark II.2 is relaxed to a requirement regarding the convergence of seconds moments (see Remark A.2 and Theorem A.2 for more details).

Theorem IV.2. *Let (X, Y) be a pair of RVs and let $\{Y_n\}_{n=1}^\infty$ be a sequence of RVs such that*

- 1) $\|X\|_{\text{RV}} < \infty$;
- 2) $(X, Y_n) \xrightarrow{d} (X, Y)$;
- 3) $X \xrightarrow{d} Y \xrightarrow{d} Y_n$ for all $n \in \mathbb{N}$.

Then, the MMSE is continuous in distribution:

$$\lim_{n \rightarrow \infty} \text{MMSE}(X|Y_n) = \text{MMSE}(X|Y).$$

Proof: Since $\|X\|_{\text{RV}} < \infty$, the second moment of the fixed sequence $\{X\}_{n=1}^{\infty}$ trivially converges to that of X . Hence, we can apply Theorem IV.1 with $X_n = X$ to attain

$$\overline{\lim}_{n \rightarrow \infty} \text{MMSE}(X|Y_n) \leq \text{MMSE}(X|Y). \quad (8)$$

Now, since $X \xrightarrow{d} Y \xrightarrow{d} Y_n$ for all $n \in \mathbb{N}$,

$$\text{MMSE}(X|Y) \leq \text{MMSE}(X|Y_n)$$

for all $n \in \mathbb{N}$ (see Lemma A.3 for details). Consequently,

$$\text{MMSE}(X|Y) \leq \underline{\lim}_{n \rightarrow \infty} \text{MMSE}(X|Y_n). \quad (9)$$

Combining (8) and (9) proves the desired result. ■

V. LMMSE CONTINUITY IN DISTRIBUTION

In this section, we treat the continuity in distribution of the LMMSE which is defined and characterized next [28, Chapter 8.4], [30, Chapter 9.1], [7, Chapter 3].

Definition V.1. The LMMSE in (linearly) estimating an RV $X \in \mathbb{R}^k$ from an RV $Y \in \mathbb{R}^m$ such that $\|X\|_{\text{RV}}, \|Y\|_{\text{RV}} < \infty$, is defined as

$$\text{LMMSE}(X|Y) \triangleq \inf \|X - (AY + b)\|_{\text{RV}}^2,$$

where the infimum is over all deterministic vectors $b \in \mathbb{R}^k$ and deterministic matrices $A \in \mathbb{R}^{k \times m}$.

Theorem V.1. Let X and Y be two RVs such that $\|X\|_{\text{RV}}, \|Y\|_{\text{RV}} < \infty$. Denote

$$\begin{aligned} \eta_X &\triangleq \mathbb{E}[X], & C_X &\triangleq \mathbb{E}[(X - \eta_X)(X - \eta_X)^T], \\ \eta_Y &\triangleq \mathbb{E}[Y], & C_Y &\triangleq \mathbb{E}[(Y - \eta_Y)(Y - \eta_Y)^T], \\ & & C_{X,Y} &\triangleq \mathbb{E}[(X - \eta_X)(Y - \eta_Y)^T], \end{aligned}$$

and assume that C_Y is invertible.¹ Then, the LMMSE estimate \hat{X} of X from Y is given by

$$\hat{X} = \eta_X + C_{X,Y} C_Y^{-1} (Y - \eta_Y),$$

i.e., $A = C_{X,Y} C_Y^{-1}$ and $b = \eta_X - C_{X,Y} C_Y^{-1} \eta_Y$ in Definition V.1. The corresponding LMMSE is given as

$$\begin{aligned} \text{LMMSE}(X|Y) &= \|X\|_{\text{RV}}^2 - \|\hat{X}\|_{\text{RV}}^2 \\ &= \text{trace}\{C_X - C_{X,Y} C_Y^{-1} C_{X,Y}^T\}. \end{aligned}$$

The following theorem establishes the continuity of the LMMSE in distribution under adequate conditions.

Theorem V.2. Let (X, Y) be a pair of RVs and let $\{(X_n, Y_n)\}_{n=1}^{\infty}$ be a sequence of pairs of RVs such that

$$1) (X_n, Y_n) \xrightarrow[n \rightarrow \infty]{d} (X, Y);$$

¹If C_Y is not invertible, this means that the entries of Y are linearly dependent a.s. Hence, to attain the LMMSE estimator, one may remove all the linearly dependent entries and estimate from the remaining entries without loss of performance.

$$2) \lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2] \text{ and } \lim_{n \rightarrow \infty} \mathbb{E}[Y_n^2] = \mathbb{E}[Y^2].$$

3) C_Y is invertible.

Then, the LMMSE is continuous in distribution:

$$\lim_{n \rightarrow \infty} \text{LMMSE}(X_n|Y_n) = \text{LMMSE}(X|Y).$$

Requirement 2 guarantees the convergence of the means (see Theorem A.1), and all the second order statistics by the Cauchy–Schwarz inequality. Since the LMMSE depends only on the second order statistics by Theorem V.1, this suffices to guarantee the continuity of the LMMSE; for a formal proof see Appendix F.

We next review examples II.1 and II.2 and introduce a new example to demonstrate the necessity of the requirements of Theorem V.2.

Example V.1. Consider the setting of Example II.1. Note that all the MMSE estimators in this example are linear, meaning that the MMSEs coincide with their corresponding LMMSEs. This demonstrates, in turn, the necessity of the convergence of the second moment of $\{Y_n\}_{n=1}^{\infty}$ to that of Y in requirement 2 of Theorem V.2.

Example V.2. Let X be some random variable with zero mean and unit variance. Set $Y = X$, $X_n = X$ for all $n \in \mathbb{N}$, and

$$Y_n = \begin{cases} \sqrt{n}, & \text{w.p. } \frac{1}{2n} \\ -\sqrt{n}, & \text{w.p. } \frac{1}{2n} \\ X, & \text{w.p. } 1 - \frac{1}{n}. \end{cases}$$

Clearly, requirements 1 and 3 of Theorem V.2 hold for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \lim_{n \rightarrow \infty} 1 = 1 = \mathbb{E}[X^2].$$

However,

$$\lim_{n \rightarrow \infty} \mathbb{E}[Y_n^2] = 2 > 1 = \mathbb{E}[Y^2].$$

Hence the second part of requirement 2 of Theorem V.2 is violated. Indeed, using the standard formula for the LMMSE (see Theorem V.1) yields

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{LMMSE}(X_n|Y_n) &= \lim_{n \rightarrow \infty} \left(\mathbb{E}[X_n^2] - \frac{\mathbb{E}[X_n Y_n]}{\mathbb{E}[Y_n^2]} Y_n \right) \\ &= 1 - \frac{1}{2} > 0 = \text{LMMSE}(X|Y), \end{aligned}$$

which demonstrates the necessity of the second part of requirement 2 in Theorem V.2.

Example V.3. Consider the setting of Example II.2. By Theorem V.2, the LMMSE is continuous, as long as $\text{Var}(Y) > 0$. The discrepancy between the continuity of the LMMSE and the discontinuity of the MMSE stems from the linearity constraint of the LMMSE estimator, as the perfect recovery of X from Y_n is non-linear. However, for $Y = 0$ and X such that $\text{Var}(X) > 0$:

$$\text{LMMSE}(X_n|Y_n) = 0 \quad \forall n \in \mathbb{N},$$

$$\text{LMMSE}(X|Y) = \text{Var}(X) > 0.$$

This, in turn, demonstrates the necessity of requirement 3 in Theorem V.2, which is violated in this case.

VI. DISCUSSION AND FUTURE WORK

This work focused on bridging the gap between the perception of practitioners of the MMSE being robust and the claim of the theoreticians of the MMSE being discontinuous in general.

By introducing a Markov restriction (1) between the nominal parameter, the nominal measurement, and the converging measurement, we proved that MMSE is in fact continuous assuming converging second moments. Such a restriction may be of interest beyond MMSE estimation, e.g., in other inference problems.

Assuming converging second moments, we further established results on the upper-semicontinuity in distribution and continuity for MMSE in estimating a parameter from a converging sequence of channels under an *individual* statistical degradedness assumption of each converging channel with respect to the limit channel.

Finally, we proved that the MMSE under linear estimation is continuous in distribution assuming again converging second moments.

It would be interesting to explore under what other conditions the MMSE is continuous. Following [23], [26], [27], [32], It would be interesting to extend the results of our work to other cost functions [3, Chapter 4], [33].

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APPENDIX A

BACKGROUND ON STOCHASTIC CONVERGENCES AND STOCHASTIC DEGRADEDNESS / GARBLING

A. Stochastic Convergences

We first present four standard definitions of stochastic convergence [31, Chapter 5], [34, Chapter 2].

Definition A.1 (stochastic convergences). Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random vectors (RVs) in \mathbb{R}^k , and let X be an RV, defined on the same probability space (Ω, \mathcal{F}, P) . Then,

- 1) *Convergence in distribution*. $\{X_n\}_{n=1}^{\infty}$ converges in distribution to X if

$$\lim_{n \rightarrow \infty} P(X_n \leq x) = P(X \leq x)$$

for every $x \in \mathbb{R}^k$ at which the cumulative distribution function of X , $P(X \leq x)$, is continuous at x . We denote this convergence by $X_n \xrightarrow[n \rightarrow \infty]{d} X$.

- 2) *Convergence in probability*. $\{X_n\}_{n=1}^{\infty}$ converges in probability to X if, for all $\epsilon > 0$,²

$$\lim_{n \rightarrow \infty} P(\|X_n - X\| > \epsilon) = 0.$$

²Other metrics between X_n and X can be used as well.

We denote this convergence by $X_n \xrightarrow[n \rightarrow \infty]{p} X$.

- 3) *Almost-sure convergence*. $\{X_n\}_{n=1}^{\infty}$ converges almost surely (a.s.) to X if

$$P\left(\lim_{n \rightarrow \infty} X_n = X\right) = 1$$

or, equivalently, if²

$$P\left(\lim_{n \rightarrow \infty} \|X_n - X\| = 0\right) = 1.$$

We denote this convergence by $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$.

- 4) *Mean-square (m.s.) convergence*. $\{X_n\}_{n=1}^{\infty}$ converges in square mean to X if $\|X\|_{\text{RV}} < \infty$, $\|X_n\|_{\text{RV}} < \infty$ for all $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \|X_n - X\|_{\text{RV}} = 0.$$

We denote this convergence by $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X$.

The following is an alternative definition of convergence in distribution, also known as *convergence in law* or *weak convergence* [31, Chapter 5, Definition 1.5].

Definition A.2. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs in \mathbb{R}^k , and let X be an RV in \mathbb{R}^k . Then, $\{X_n\}_{n=1}^{\infty}$ converges in distribution to X ($X_n \xrightarrow[n \rightarrow \infty]{d} X$) if

$$\lim_{n \rightarrow \infty} E[f(X_n)] = E[f(X)]$$

for all bounded and continuous functions $f : \mathbb{R}^k \rightarrow \mathbb{R}$.

The equivalence of the two definitions for convergence in distribution is often presented as part of the *Portmanteau lemma* [34, Chapter 2], [35, Chapters 2 and 3].

Remark A.1. While convergences in probability, in m.s., and a.s. require the RVs in $\{X_n\}_{n=1}^{\infty}$ and X to be defined on the same probability space, this is not necessary for convergence in distribution.

The proof of the following lemma is available in the appendix.

Lemma A.1. Let $k \in \mathbb{N}$. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random vectors in \mathbb{R}^k , and let X be a random vector in \mathbb{R}^k , defined on the same probability space (Ω, \mathcal{F}, P) . Then,

- a) $X_n \xrightarrow[n \rightarrow \infty]{p} X \Leftrightarrow X_n[i] \xrightarrow[n \rightarrow \infty]{p} X[i] \quad \forall i \in \{1, 2, \dots, k\};$
- b) $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X \Leftrightarrow X_n[i] \xrightarrow[n \rightarrow \infty]{a.s.} X[i] \quad \forall i \in \{1, 2, \dots, k\};$
- c) $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X \Leftrightarrow X_n[i] \xrightarrow[n \rightarrow \infty]{m.s.} X[i] \quad \forall i \in \{1, 2, \dots, k\};$
- d) $X_n \xrightarrow[n \rightarrow \infty]{d} X \Rightarrow X_n[i] \xrightarrow[n \rightarrow \infty]{d} X[i] \quad \forall i \in \{1, 2, \dots, k\}.$

The results in the following lemma and theorem are well known; see [31, Chapter 5, Theorems 3.1 and 5.4].

Lemma A.2. Let $k \in \mathbb{N}$. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs in \mathbb{R}^k , and let X be an RV in \mathbb{R}^k , defined on the same probability space (Ω, \mathcal{F}, P) . Then, the following relations hold:

- 1) $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X \Rightarrow X_n \xrightarrow[n \rightarrow \infty]{p} X;$

- 2) $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X \Rightarrow X_n \xrightarrow[n \rightarrow \infty]{p} X$;
 3) $X_n \xrightarrow[n \rightarrow \infty]{p} X \Rightarrow X_n \xrightarrow[n \rightarrow \infty]{d} X$.

Lemma A.2 states that m.s. convergence guarantees convergence in probability; the opposite direction does not hold in general. However, under uniform integrability, defined next, the opposite direction holds as well.

Definition A.3 (uniform integrability [31, Chapter 4]). A sequence $\{X_n \in \mathbb{R} | n \in \mathbb{N}\}$ of random variables is said to be *uniformly integrable (u.i.)* if for every $\epsilon > 0$, there exists a constant $a > 0$ such that

$$\mathbb{E}[\mathbf{1}\{|X_n| > a\} \cdot |X_n|] \leq \epsilon \quad \forall n \in \mathbb{N},$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. A sequence of RVs $\{X_n \in \mathbb{R}^k | n \in \mathbb{N}\}$ for $k \in \mathbb{N}$ is said to be u.i. if $\{X_n[i] | n \in \mathbb{N}\}$ is u.i. for all $i \in \{1, 2, \dots, k\}$.

Remark A.2. An almost-surely bounded sequence $\{X_n\}_{n=1}^{\infty}$, viz. a sequence satisfying $\mathbb{P}(\|X_n\| \leq m) = 1$ for all $n \in \mathbb{N}$ for some $m \in \mathbb{R}$, is trivially u.i.

Theorem A.1 (see [35, Section 3.4]). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs, and let X be an RV, such that $X_n \xrightarrow[n \rightarrow \infty]{d} X$. Then, the following statements are equivalent:*

- $\|X\|_{\text{RV}} < \infty$, $\|X_n\|_{\text{RV}} < \infty$ for all $n \in \mathbb{N}$, and

$$\lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2];$$

- $\{X_n^2\}_{n=1}^{\infty}$ is u.i.

Furthermore, if one of the above statements holds, then $\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X]$.

Theorem A.2 (see [31, Chapter 5, Section 5.2, Th. 5.4]). *Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs, and let X be an RV. Then, the following statements are equivalent:*

- $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X$;
- $X_n \xrightarrow[n \rightarrow \infty]{p} X$, $\|X\|_{\text{RV}} < \infty$, $\|X_n\|_{\text{RV}} < \infty$ for all $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2]$;
- $X_n \xrightarrow[n \rightarrow \infty]{p} X$ and $\{X_n^2\}_{n=1}^{\infty}$ is u.i.

Furthermore, if one of the above statements holds, then $\lim_{n \rightarrow \infty} \mathbb{E}[X_n] = \mathbb{E}[X]$.

B. Stochastic Degradedness / Garbling

The following notion of *stochastic degradedness* or *garbling* and theorem will be used in the derivation of continuity in distribution results in Section IV. We define this notion in terms of RVs rather than in the more common terms of conditional distributions [25, Definition 7.3.1].

Definition A.4. Let (X_1, Y_1) and (X_2, Y_2) be two pairs of RVs. We say that (X_2, Y_2) is *stochastically degraded* or *garbled* with respect to (X_1, Y_1) if

$$X_1 \stackrel{d}{=} X_2$$

and there exist $\tilde{X}, \tilde{Y}_1, \tilde{Y}_2$ such that

$$\begin{aligned} (X_i, Y_1) &\stackrel{d}{=} (\tilde{X}, \tilde{Y}_1), \\ (X_i, Y_2) &\stackrel{d}{=} (\tilde{X}, \tilde{Y}_2), \\ \tilde{X} &\rightarrow \tilde{Y}_1 \rightarrow \tilde{Y}_2. \end{aligned}$$

This definition means that we can view these two pairs as two channels with the same input \tilde{X} where the channel to the first output \tilde{Y}_1 is more informative than that to the second output \tilde{Y}_2 . Since we are interested only in the marginal distributions of the pairs (X_1, Y_1) and of (X_2, Y_2) but not the joint distribution of the quadruple, we can specialize Definition A.4 to the following.

Definition A.5. Let X, Y_1, Y_2 be three RVs. We will say that, given X, Y_2 is *stochastically degraded* or *garbled* with respect to Y_1 if there exists an RV \tilde{Y}_1 such that:

$$\begin{aligned} (X, \tilde{Y}_1) &\stackrel{d}{=} (X, Y_1), \\ X &\rightarrow \tilde{Y}_1 \rightarrow Y_2. \end{aligned}$$

We denote this by $X \xrightarrow{d} Y_1 \xrightarrow{d} Y_2$.

The following simple result can be viewed as a specialization of Blackwell's informativeness theorem [25, Chapter 7.3.1] for MMSEs; see, e.g., [22, Theorem 11] for a proof.³

Lemma A.3. *Let $X \xrightarrow{d} Y_1 \xrightarrow{d} Y_2$. Then,*

$$\text{MMSE}(X|Y_1) \leq \text{MMSE}(X|Y_2),$$

with equality if and only if $\mathbb{E}[X|Y_1] = \mathbb{E}[X|Y_2]$ a.s.

APPENDIX B PROOF OF LEMMA A.1

a) See [34, Theorem 2.7].

b) Define the events $A_\ell = \left\{ \lim_{n \rightarrow \infty} X_n[\ell] = X_n[\ell] \right\}$ for all $\ell \in \{1, 2, \dots, k\}$ and $A = \left\{ \lim_{n \rightarrow \infty} X_n = X \right\}$. Clearly, $A = \bigcap_{\ell=1}^k A_\ell$.

Assume first that $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$ and set some $i \in \{1, 2, \dots, k\}$. Then,

$$1 = \mathbb{P}(A) = \mathbb{P}\left(\bigcap_{\ell=1}^k A_\ell\right) \leq \mathbb{P}(A_i) \leq 1.$$

Thus, by the squeeze theorem, $X_n[i] \xrightarrow[n \rightarrow \infty]{a.s.} X[i]$ for all $i \in \{1, 2, \dots, k\}$.

Now assume $X_n[i] \xrightarrow[n \rightarrow \infty]{p} X[i] \quad \forall i \in \{1, 2, \dots, k\}$. Then,

$$1 \geq \mathbb{P}(A) = \mathbb{P}\left(\bigcap_{\ell=1}^k A_\ell\right) = 1 - \mathbb{P}\left(\bigcup_{\ell=1}^k A_\ell^c\right) \geq 1 - \sum_{\ell=1}^k \mathbb{P}(A_\ell^c) \geq 1.$$

Thus, by the squeeze theorem, $X_n \xrightarrow[n \rightarrow \infty]{a.s.} X$.

c) The result immediately follows by noting that

$$\|X - X_n\|_{\text{RV}}^2 = \sum_{i=1}^k \|X[i] - X_n[i]\|_{\text{RV}}^2.$$

³The proof of [22, Theorem 11] assumed $X \rightarrow Y_1 \rightarrow Y_2$ but the result and the proof are intact for $X \xrightarrow{d} Y_1 \xrightarrow{d} Y_2$.

d) Follows immediately from the definition of convergence in distribution. ■

APPENDIX C
PROOF OF LEMMA III.2

Assume that X (and hence also X_n) is of dimension $k \in \mathbb{N}$, and Y (and hence also Y_n) is of dimension $m \in \mathbb{N}$.

Define the functions

$$g_n(y) \triangleq \mathbb{E}[X|Y_n = y], \quad (13a)$$

$$g(y) \triangleq \mathbb{E}[X|Y = y]. \quad (13b)$$

Denote the i -th element (scalar-valued function) of the vector-valued function g_n by $g_n[i]$, and, similarly, the i -th element (scalar-valued function) of the vector-valued function g by $g[i]$.

Since $X \rightarrow Y \rightarrow Y_n$ for all $n \in \mathbb{N}$,

$$\mathbb{E}[X|Y, Y_n] = \mathbb{E}[X|Y] \quad (14)$$

a.s. for all $n \in \mathbb{N}$. Furthermore,

$$g_n(Y_n) \triangleq \mathbb{E}[X|Y_n] \quad (15a)$$

$$= \mathbb{E}[\mathbb{E}[X|Y, Y_n]|Y_n] \quad (15b)$$

$$= \mathbb{E}[\mathbb{E}[X|Y]|Y_n] \quad (15c)$$

$$= \mathbb{E}[g(Y)|Y_n], \quad (15d)$$

where (15a) follows from (13b), (15b) follows from the law of total expectation, (15c) follows from (14), and (15d) follows from (13a). Equation (15) means that g_n is the MMSE estimator of $g(Y)$ given Y_n . In particular, $g_n[i]$ is the MMSE estimator of $g[i](Y)$ given Y_n .

The remainder of the proof follows similar steps to those in [22, Appendix C]. Denote by $L^2(\mathbb{R}^k)$ the set of measurable functions that are square integrable with respect to the probability measure of Y , i.e., the set of measurable functions f that satisfy $\|f(Y)\|_{\text{RV}} < \infty$. Denote by $C_c(\mathbb{R}^k) \subset L^2(\mathbb{R}^k)$ the space of compactly-supported continuous functions (and hence also bounded) on \mathbb{R}^k . Clearly $g[i] \in L^2(\mathbb{R}^k)$ for all $i \in \{1, 2, \dots, k\}$ since

$$\|g(Y)\|_{\text{RV}} \stackrel{(a)}{=} \|\mathbb{E}[X|Y]\|_{\text{RV}} \stackrel{(b)}{\leq} \|X\|_{\text{RV}} \stackrel{(c)}{<} \infty,$$

where (a) holds by the definition of g (13a), (b) follows from (2b) in Theorem II.1 and the non-negativity of the MMSE, and (c) holds by the lemma assumption.

Moreover, since $C_c(\mathbb{R}^k)$ is dense in $L^2(\mathbb{R}^k)$ [36, Theorem 3.14], for any $i \in \{1, 2, \dots, k\}$ and any $\epsilon > 0$, there exists a function

$$\hat{g}[i] \in C_c(\mathbb{R}^k),$$

such that

$$\|g[i](Y) - \hat{g}[i](Y)\|_{\text{RV}} < \epsilon. \quad (16)$$

Let $i \in \{1, 2, \dots, k\}$ and let $\epsilon > 0$, however small. Then, there exists $\hat{g}[i] \in C_c(\mathbb{R}^k)$ that satisfies (16). Then, there exists $n_i \in \mathbb{N}$ such that, for all $n > n_i$,

$$\|g[i](Y) - g_n[i](Y_n)\|_{\text{RV}} \leq \|g[i](Y) - \hat{g}[i](Y_n)\|_{\text{RV}} \quad (17a)$$

$$\leq \|g[i](Y) - \hat{g}[i](Y)\|_{\text{RV}} + \|\hat{g}[i](Y) - \hat{g}[i](Y_n)\|_{\text{RV}} \quad (17b)$$

$$< 2\epsilon, \quad (17c)$$

where (17a) follows from g_n being the MMSE estimator of $g(Y)$ given Y_n (15); (17b) follows from the triangle (Minkowski) inequality [31, Chapter 3.1, Theorem 2.6]; (17c) holds by noting that the first term in (17b) is bounded as in (16). To bound the second term in (17b) by ϵ for a large enough n_i , note first that

$$\hat{g}[i](Y_n) \xrightarrow[n \rightarrow \infty]{p} \hat{g}[i](Y)$$

by the continuous mapping theorem [31, Chapter 5.10, Theorem 10.3].

Consequently,

$$\hat{g}[i](Y_n) \xrightarrow[n \rightarrow \infty]{d} \hat{g}[i](Y)$$

by part 3 of Lemma A.2. Since $\hat{g}[i] \in C_c(\mathbb{R}^k)$ is continuous and bounded, so is $\hat{g}[i]^2$. Hence,

$$\lim_{n \rightarrow \infty} \mathbb{E}[(\hat{g}[i](Y_n))^2] = \mathbb{E}[(\hat{g}[i](Y))^2]$$

by Definition A.2. Thus, by Theorem A.2,

$$\hat{g}[i](Y_n) \xrightarrow[n \rightarrow \infty]{m.s.} \hat{g}[i](Y), \quad (18)$$

Set $n_0 = \max_{i \in \{1, 2, \dots, k\}} n_i$. Then, by summing (17) over all $i \in \{1, 2, \dots, k\}$, we obtain

$$\|g(Y) - g_n(Y_n)\|_{\text{RV}} < 2\sqrt{k}\epsilon$$

for all $n > n_0$, which proves the desired result. ■

APPENDIX D
PROOF OF THEOREM IV.1

To prove Theorem IV.1, we will use Skorokhod's representation theorem [35, Chapter 1, Section 6], stated below.

Theorem D.1. *Let X be an RV and let $\{X_n | n \in \mathbb{N}\}$ be a sequence of RVs such that*

$$X_n \xrightarrow[n \rightarrow \infty]{d} X.$$

Then, there exists a sequence of RVs $\{\tilde{X}_n | n \in \mathbb{N}\}$, all defined on the same probability space, such that

$$\tilde{X}_n \stackrel{d}{=} X_n \quad \forall n \in \mathbb{N},$$

$$\tilde{X} \stackrel{d}{=} X,$$

$$\tilde{X}_n \xrightarrow[n \rightarrow \infty]{a.s.} \tilde{X}.$$

Proof of Theorem IV.1: Define g and g_n , and their elements $g[i]$ and $g_n[i]$ as in the proof of Lemma III.2. Set $\epsilon > 0$, however small.

By Theorem D.1, there exist $\tilde{X}, \tilde{Y}, \{\tilde{X}_n\}_{n=1}^\infty$, and $\{\tilde{Y}_n\}_{n=1}^\infty$ that satisfy:

$$(\tilde{X}_n, \tilde{Y}_n) \stackrel{d}{=} (X_n, Y_n) \quad \forall n \in \mathbb{N}, \quad (20a)$$

$$(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (X, Y), \quad (20b)$$

$$(\tilde{X}_n, \tilde{Y}_n) \xrightarrow[n \rightarrow \infty]{a.s.} (\tilde{X}, \tilde{Y}). \quad (20c)$$

Furthermore, since

$$\lim_{n \rightarrow \infty} \mathbb{E}[\tilde{X}_n^2] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n^2] = \mathbb{E}[X^2] = \mathbb{E}[\tilde{X}^2],$$

$\tilde{X}_n \xrightarrow[n \rightarrow \infty]{m.s.} \tilde{X}$ by Theorem A.2. Hence, by the definition of m.s. convergence (part 4 of Definition A.1), there exists $n_i \in \mathbb{N}$ such that, for all $n > n_i$,

$$\|\tilde{X}_n[i] - \tilde{X}[i]\|_{\text{RV}} < \epsilon.$$

As in the proof of Lemma III.2, for any $i \in \{1, 2, \dots, k\}$, there exists $\hat{g}[i] \in C_c(\mathbb{R}^k)$ that satisfies

$$\|g[i](\tilde{Y}) - \hat{g}[i](\tilde{Y})\|_{\text{RV}} < \epsilon. \quad (21)$$

Further, following the steps in the proof of (18) in Lemma III.2,

$$\hat{g}[i](\tilde{Y}_n) \xrightarrow[n \rightarrow \infty]{m.s.} \hat{g}[i](\tilde{Y}) \quad (22)$$

holds for all $i \in \{1, 2, \dots, k\}$. To wit, for any $i \in \{1, 2, \dots, k\}$, there exists $\ell_i \in \mathbb{N}$, such that

$$\|\hat{g}[i](\tilde{Y}_n) - \hat{g}[i](\tilde{Y})\|_{\text{RV}} < \epsilon \quad (23)$$

holds for all $n < \ell_i$.

We are now ready to prove the desired result. Set some $i \in \{1, 2, \dots, k\}$ and $t_i = \max(n_i, \ell_i)$. Then, for all $n > t_i$,

$$\sqrt{\text{MMSE}(X_n[i]|Y_n)} = \sqrt{\text{MMSE}(\tilde{X}_n[i]|\tilde{Y}_n)} \quad (24a)$$

$$= \|\tilde{X}_n[i] - g_n[i](\tilde{Y}_n)\|_{\text{RV}} \quad (24b)$$

$$\leq \|\tilde{X}_n[i] - \hat{g}[i](\tilde{Y}_n)\|_{\text{RV}} \quad (24c)$$

$$\leq \|\hat{g}[i](\tilde{Y}) - \hat{g}[i](\tilde{Y}_n)\|_{\text{RV}} + \|g[i](\tilde{Y}) - \hat{g}[i](\tilde{Y})\|_{\text{RV}} + \|\tilde{X}[i] - g[i](\tilde{Y})\|_{\text{RV}} + \|\tilde{X}_n[i] - \tilde{X}[i]\|_{\text{RV}} \quad (24d)$$

$$< \|\tilde{X}[i] - g[i](\tilde{Y})\|_{\text{RV}} + 3\epsilon \quad (24e)$$

$$= \sqrt{\text{MMSE}(\tilde{X}[i]|\tilde{Y})} + 3\epsilon \quad (24f)$$

$$= \sqrt{\text{MMSE}(X[i]|Y)} + 3\epsilon, \quad (24g)$$

where (24a) follows from (20a), (24b) and (24f) follow from Theorem 2a, (24c) follows from g_n being the MMSE estimator of $g(\tilde{Y})$ given \tilde{Y}_n (15), (24d) holds by the triangle (Minkowski) inequality, (24e) follows from (21)–(23), and (24g) follows from (20b).

Since (24) holds for any $\epsilon > 0$, for a sufficiently large t_i ,

$$\overline{\lim}_{n \rightarrow \infty} \text{MMSE}(X_n[i]|Y_n) \leq \text{MMSE}(X[i]|Y). \quad (25)$$

Consequently, the desired result follows:

$$\overline{\lim}_{n \rightarrow \infty} \text{MMSE}(X_n|Y_n) = \overline{\lim}_{n \rightarrow \infty} \sum_{i=1}^k \text{MMSE}(X_n[i]|Y_n) \quad (26a)$$

$$\leq \sum_{i=1}^k \overline{\lim}_{n \rightarrow \infty} \text{MMSE}(X_n[i]|Y_n) \quad (26b)$$

$$\leq \sum_{i=1}^k \text{MMSE}(X[i]|Y) \quad (26c)$$

$$= \text{MMSE}(X|Y), \quad (26d)$$

where (26a) and (26d) follow from Definition II.1, (26b) follows from limit-superior arithmetics, and (26c) follows from (25).

APPENDIX E

PROOFS OF COROLLARIES III.1 AND III.2

Proof of Corollary III.1: Let $\{\gamma \in \mathbb{R}\}_{n=1}^{\infty}$ and $\{\lambda \in \mathbb{R}\}_{n=1}^{\infty}$ be some sequences that converge to zero. Denote $X_n = X + \gamma_n N$ and $Y_n = Y + \lambda_n M$ for $n \in \mathbb{N}$.

Since $\|X\|_{\text{RV}}, \|N\|_{\text{RV}} < \infty$,

$$\|X_n\|_{\text{RV}} \leq \|X\|_{\text{RV}} + |\gamma_n| \|N\|_{\text{RV}} < \infty \quad \forall n \in \mathbb{N},$$

where the inequality follows from the triangle (Minkowski) inequality. Further,

$$\lim_{n \rightarrow \infty} \|X_n - X\|_{\text{RV}} = \lim_{n \rightarrow \infty} \|\gamma_n N\|_{\text{RV}} = \lim_{n \rightarrow \infty} |\gamma_n| \cdot \|N\|_{\text{RV}} = 0,$$

which means that $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X$ by Definition A.1. Furthermore, (6) holds by Theorem A.2.

Set some $\epsilon > 0$. Then,

$$\lim_{n \rightarrow \infty} \mathbb{P}(\|Y_n - Y\| > \epsilon) = \lim_{n \rightarrow \infty} \mathbb{P}\left(\|M\| > \frac{\epsilon}{\lambda_n}\right) = 0,$$

meaning that $Y_n \xrightarrow[n \rightarrow \infty]{p} Y$ by Definition A.1. Hence,

$$(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{p} (X, Y)$$

by Lemmata A.1 and A.2.

Since M is independent of (X, Y) , the Markov condition (1) $X \rightarrow Y \rightarrow Y_n$ holds for all $n \in \mathbb{N}$.

Thus, by Theorem III.1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{MMSE}(X + \gamma_n N | Y + \lambda_n M) &= \lim_{n \rightarrow \infty} \text{MMSE}(X_n | Y_n) \\ &= \text{MMSE}(X | Y). \end{aligned}$$

Since the above holds for any sequence $\{(\lambda_n, \gamma_n)\}_{n=1}^{\infty}$ that satisfies $(\lambda_n, \gamma_n) \xrightarrow[n \rightarrow \infty]{} (0, 0)$, the desired result (7) follows. ■

Proof of Corollary III.2: Let $\{\gamma \in \mathbb{R}\}_{n=1}^{\infty}$ and $\{\lambda \in \mathbb{R}\}_{n=1}^{\infty}$ be some sequences that converge to zero. Denote $X_n = \lfloor X \rfloor_{\gamma_n}$ and $Y_n = \lfloor Y \rfloor_{\lambda_n}$.

Since $X_n - X \in (-\gamma_n, \gamma_n)^k$, $\|X_n - X\|_{\text{RV}} < \sqrt{k} |\gamma_n|$. Since $\lim_{n \rightarrow \infty} \gamma_n = 0$, this implies

$$\lim_{n \rightarrow \infty} \|X_n - X\|_{\text{RV}} = 0.$$

Since $\|Y\|_{\text{RV}} < \infty$, by the triangle (Minkowski) inequality,

$$|\|X_n\|_{\text{RV}} - \|X\|_{\text{RV}}| \leq \|X_n - X\|_{\text{RV}}.$$

Thus, $\|X_n\|_{\text{RV}} < \infty$ for all $n \in \mathbb{N}$ as well. Hence, $X_n \xrightarrow[n \rightarrow \infty]{m.s.} X$ by Definition A.1.

Similarly, since $Y_n - Y \in (-\lambda_n, \lambda_n)^k$, $\|Y_n - Y\| < \sqrt{k} |\lambda_n|$. Consequently,

$$\mathbb{P}(\|Y_n - Y\| > \epsilon) = 0,$$

for $\sqrt{k} |\lambda_n| < \epsilon$. Since $\lim_{n \rightarrow \infty} \lambda_n = 0$, $Y_n \xrightarrow[n \rightarrow \infty]{p} Y$ by Definition A.1. Hence, $(X_n, Y_n) \xrightarrow[n \rightarrow \infty]{p} (X, Y)$ by Lemmata A.1 and A.2.

Since $X_n = \lfloor Y \rfloor_{\lambda_n}$ is a deterministic function of Y , the Markov condition (1) $X \rightarrow Y \rightarrow Y_n$ holds for all $n \in \mathbb{N}$. ■

Thus, by Theorem III.1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{MMSE}(\lfloor X \rfloor_{\gamma_n} | \lfloor Y \rfloor_{\lambda_n}) &= \lim_{n \rightarrow \infty} \text{MMSE}(X_n | Y_n) \\ &= \text{MMSE}(X | Y). \end{aligned}$$

Since, the above holds for any sequence $\{(\lambda_n, \gamma_n)\}_{n=1}^{\infty}$ that satisfies $(\lambda_n, \gamma_n) \xrightarrow{n \rightarrow \infty} (0, 0)$, the desired result (7) follows. ■

APPENDIX F PROOF OF THEOREM V.2

We will first prove the following lemma.

Lemma F.1. *Let (X, Y) be a pair of random variables and let $\{(X_n, Y_n)\}_{n=1}^{\infty}$ be a sequence of pairs of random variables such that*

$$\lim_{n \rightarrow \infty} \text{E}[X_n^2] = \text{E}[X^2], \quad \lim_{n \rightarrow \infty} \text{E}[Y_n^2] = \text{E}[Y^2].$$

Then,

$$\lim_{n \rightarrow \infty} \text{Cov}(X_n, Y_n) = \text{Cov}(X, Y),$$

where $\text{Cov}(A, B) \triangleq \text{E}[(A - \text{E}[A])(B - \text{E}[B])]$ denotes the covariance between A and B .

Proof: By Skorokhod's theorem (Theorem D.1), there exist $\tilde{X}, \tilde{Y}, \{(\tilde{X}_n, \tilde{Y}_n)\}_{n=1}^{\infty}$ such that

$$(\tilde{X}_n, \tilde{Y}_n) \stackrel{d}{=} (X_n, Y_n) \quad \forall n \in \mathbb{N}, \quad (27a)$$

$$(\tilde{X}, \tilde{Y}) \stackrel{d}{=} (X, Y), \quad (27b)$$

$$(\tilde{X}_n, \tilde{Y}_n) \xrightarrow[n \rightarrow \infty]{a.s.} (\tilde{X}, \tilde{Y}).$$

Therefore,

$$\begin{aligned} &|\text{Cov}(X_n, Y_n) - \text{Cov}(X, Y)| \\ &= |\text{Cov}(\tilde{X}_n, \tilde{Y}_n) - \text{Cov}(\tilde{X}, \tilde{Y})| \end{aligned} \quad (28a)$$

$$= |\text{Cov}(\tilde{X}, \tilde{Y}_n - \tilde{Y}) - \text{Cov}(\tilde{X} - \tilde{X}_n, \tilde{Y}_n)| \quad (28b)$$

$$\leq \sqrt{\text{Var}(\tilde{X})} \sqrt{\text{Var}(\tilde{Y}_n - \tilde{Y})} + \sqrt{\text{Var}(\tilde{X}_n - \tilde{X})} \sqrt{\text{Var}(\tilde{Y}_n)}, \quad (28c)$$

where (28a) follows from (27a) and (27b), (28b) follows from the bi-linearity of the covariance, and (28c) follows from the Cauchy–Schwarz inequality.

By Theorem A.2,

$$0 \leq \lim_{n \rightarrow \infty} \text{Var}(\tilde{X}_n - \tilde{X}) \leq \lim_{n \rightarrow \infty} \text{E}[(\tilde{X}_n - \tilde{X})^2] = 0, \quad (29)$$

(and similarly for \tilde{Y}_n). Thus, by the squeeze theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Var}(\tilde{X}_n - \tilde{X}) &= 0, \\ \lim_{n \rightarrow \infty} \text{Var}(\tilde{Y}_n - \tilde{Y}) &= 0. \end{aligned} \quad (30)$$

Further, by Theorem A.2,

$$\lim_{n \rightarrow \infty} \text{Var}(\tilde{Y}_n) = \text{Var}(\tilde{Y}) < \infty.$$

Hence, the desired result follows from (28)–(30) and the squeeze theorem. ■

We are now ready to prove Theorem V.2.

Proof of Theorem V.2: Since C_Y is invertible, its determinant $\det\{C_Y\} \neq 0$ and Theorem V.1 is applicable.

By Theorem A.1,

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{E}[X_n] &= \text{E}[X], \\ \lim_{n \rightarrow \infty} \text{E}[Y_n] &= \text{E}[Y]. \end{aligned} \quad (31)$$

By Lemma F.1,

$$\begin{aligned} \lim_{n \rightarrow \infty} C_{Y_n} &= C_Y, \\ \lim_{n \rightarrow \infty} C_{X_n, Y_n} &= C_{X, Y}. \end{aligned} \quad (32)$$

Furthermore, since

$$C_Y^{-1} = \frac{\text{adj}\{C_Y\}}{\det\{C_Y\}},$$

we also have

$$\lim_{n \rightarrow \infty} C_{Y_n}^{-1} = C_Y^{-1}, \quad (33)$$

where $\text{adj}\{C_Y\}$ denotes the adjugate of C_Y .

The desired result then follows from the LMMSE formula in Theorem V.1, Theorem A.1 and the convergence results in (31)–(33). ■

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